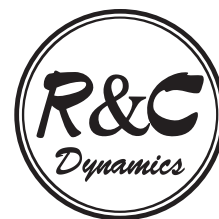


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# EULER–POISSON EQUATIONS AND INTEGRABLE CASES

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In this paper we propose a new approach to the study of integrable cases based on intensive computer methods' application. We make a new investigation of Kovalevskaya and Goryachev–Chaplygin cases of Euler–Poisson equations and obtain many new results in rigid body dynamics in absolute space. Also we present the visualization of some special particular solutions.

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## 1. Euler–Poisson equations and integrable cases

In this paper we propose a new approach to the study of the classical problem of integrable cases in rigid body dynamics. It is based on computer methods' application to both analytical and numerical investigation of the systems in question. This approach allows to obtain many new results, some of which are presented in the paper (for detailed presentation, see [9]).

Even in the analysis of integrable cases that, basically, allow complete classification of all the solutions, the computer research methods have, in some sense, started a new age. Earlier investigations of integrable systems involved mostly the analytical methods permitting to obtain the explicit quadratures and geometrical interpretations which in many cases were very artificial (consider, for example, Zhukovsky's interpretation of Kovalevskaya top's motion [12]). By contrast the combination of the ideas of topological analysis (bifurcation diagrams), stability theory, method of phase sections, and immediate computer visualization of "particularly special" solutions is quite able to demonstrate the specific character of an integrable situation and to single out the most typical properties of motion. With such investigation, it is possible to obtain a variety of new results, even in the field that seems to be thoroughly studied (for example, for the tops of Kovalevskaya and Goryachev–Chaplygin, and for Bobylev–Stekloff solution). The matter is that it is very hard to see these results in those cumbersome analytical expressions. The proofs of these facts can probably be obtained analytically as well, but only after their revealing by computer methods. Here we should especially note the analysis of motion in the absolute space.

Some specific motions of such integrable tops can probably initiate some concrete ideas, concerning their practical investigation. Let's remind, for example, that the Kovalevskaya top discovered more than a century ago still can not find any application just because its motion remains practically unknown despite of its complete solutions in elliptic functions.

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We also give certain unstable periodic solutions that generate a family of doubly-asymptotic motions, whose behavior is most complicated and looks irregular even in cases with an additional integral. Under perturbation, such solutions are first to collapse and the whole areas, filled with now “real” chaotic trajectories, appear near them in the phase space.

The computer investigations force “the revision” of many aspects of analytical investigations and help to understand their real meaning. While some analytical results, such as separation of variables, are very useful for the study of bifurcations and classical solutions, their further “development” to explicit quadratures (through  $\theta$ -functions) is practically useless. These results are collected, for example, in [21, 17], but they are more useful as exercise in differential equations than as methods of dynamics analysis.

### 1.1. A rigid body with a fixed point

*Euler–Poisson equations* describing the motion of a rigid body around a fixed point in homogeneous gravity field have the form

$$\begin{cases} \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \mu \mathbf{r} \times \boldsymbol{\gamma}, \\ \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \end{cases} \quad (1.1)$$

where  $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ ,  $\mathbf{r} = (r_1, r_2, r_3)$ ,  $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$  are, respectively, the components of the angular velocity vector, the components of the radius vector of the center of mass, and the components of the vertical unit vector in the frame of the principal axes, rigidly bound with the rigid body and passing through the point of fixation,  $\mathbf{I} = \text{diag}(I_1, I_2, I_3)$  is the tensor of inertia in relation to the point of fixation in the same axes,  $\mu = mg$  is the weight of the body (Fig. 1).

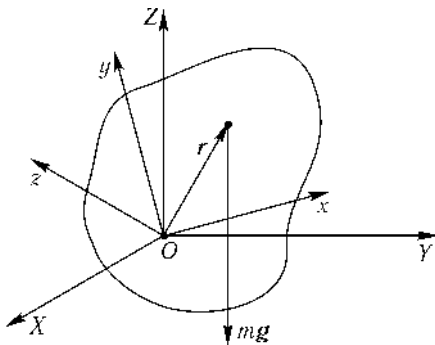


Fig. 1. A rigid body with a fixed point in gravity field.

Using the projections of the momentum vector  $\mathbf{M} = \mathbf{I}\boldsymbol{\omega}$  in the same axes, equations (1.1) can be presented in the Hamiltonian form

$$\dot{M}_i = \{M_i, H\}, \quad \dot{\gamma}_i = \{\gamma_i, H\}, \quad i = 1, 2, 3, \quad (1.2)$$

with a Poisson bracket corresponding to algebra  $e(3)$

$$\{M_i, M_j\} = -\varepsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = -\varepsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\} = 0, \quad (1.3)$$

and with the Hamiltonian equal to the full energy of body

$$H = \frac{1}{2}(\mathbf{A}\mathbf{M}, \mathbf{M}) - \mu(\mathbf{r}, \boldsymbol{\gamma}). \quad (1.4)$$

where  $\mathbf{A} = \mathbf{I}^{-1}$ .

REMARK 1. Euler (1758) already knew the equations of motion in form (1.1), he also had found the elementary case of integrability, when the rigid body moves under inertia ( $\mathbf{r} = 0$ ). The integrability of an axially symmetric top with the center of gravity on the symmetry axis was established by Lagrange and a little bit later by Poisson, the latter’s name being included in the term for the general equations (1.1).

Lie–Poisson bracket (1.3) is degenerated, it has two Casimir functions, commuting with any function of variables  $\mathbf{M}$ ,  $\boldsymbol{\gamma}$  in the structure of brackets (1.3)

$$F_1 = (\mathbf{M}, \boldsymbol{\gamma}), \quad F_2 = \boldsymbol{\gamma}^2. \quad (1.5)$$

In the vector form, equation (1.2) can be written as

$$\begin{cases} \dot{\mathbf{M}} = \mathbf{M} \times \frac{\partial H}{\partial \mathbf{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \mathbf{M}}. \end{cases} \quad (1.6)$$

The form of equations (1.2), (1.6) is induced by the Poincaré–Chetayev equations, written on group  $SO(3)$  (see [9]).

Functions  $F_1$  and  $F_2$  are integrals of equations (1.6) for any Hamiltonian function  $H$ . For Euler–Poisson equations they have a natural physical and geometrical origin. Integral  $F_1$  represents a projection of the momentum vector on the fixed vertical axis and it is referred to as *area integral* in rigid body dynamics, it represents the symmetry in relation to the rotations around the fixed vertical axis. The origin of integral  $F_2 = \text{const}$  is purely geometrical, it is equal to the squared absolute value of the vertical unit vector. For real motions the value of the constant of this integral is equal to one:  $F_2 = \gamma^2 = 1$ .

When bracket (1.3) is restricted on the common level of integrals  $F_1$  and  $F_2$ , it becomes nondegenerate, and Darboux theorem ([9]) implies that the bracket can be represented in the usual canonical form in certain symplectic coordinates. For the various purposes it is possible to use both canonical Euler variables  $(\theta, \varphi, \psi, p_\theta, p_\varphi, p_\psi)$  and Andoyer–Deprit variables  $(L, G, H, l, g, h)$ . In both cases on the symplectic leaf defined by  $p_\psi = \text{const}$  (respectively,  $H = \text{const}$ ) we obtain the canonical system with two degrees of freedom.

For Liouville integrability ([9]) of system (1.1), and system (1.6) as well, the presence of one more additional integral is necessary besides Hamiltonian (1.4), which is also a first integral of the system.

## 2. Kovalevskaya case

It is known that this integral exists in the cases of Euler, Lagrange and Kovalevskaya, and with additional restriction  $(M, \gamma) = 0$  in Goryachev–Chaplygin case. While in the first two cases the motion has been studied thoroughly enough, Kovalevskaya and Goryachev–Chaplygin cases are still poorly investigated. The additional integrals in Euler and Lagrange cases have natural physical origin. In the first case the integral is the squared absolute value of the momentum vector, in the second case the integral is the projection of this quantity on the axis of dynamical symmetry. In the case of integrability found by S.V. Kovalevskaya (1888) the additional integral has no explicit symmetrical origin. It was found almost a century after two previous integrals, and it is incomparably more complicated both from the point of view of explicit integration and for the qualitative analysis of motion.

The rigid body in this case is dynamically symmetric:  $a_1 = a_2$ , and the center of mass is situated on the equatorial plane of the inertia ellipsoid  $r_3 = 0$ . In this case relation  $\frac{a_3}{a_1} = \frac{I_1}{I_3} = 2$  is also valid. The Hamiltonian and the additional integral found by Kovalevskaya are given by:

$$\begin{aligned} H &= \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) - x\gamma_1, \\ F_3 &= \left( \frac{M_1^2 - M_2^2}{2} + x\gamma_1 \right)^2 + (M_1M_2 + x\gamma_2)^2 = k^2, \end{aligned} \quad (2.1)$$

where the components of the radius vector of the center of mass are  $\mathbf{r} = (x, 0, 0)$  and the weight is equal to  $\mu = 1$  without loss of generality.

### 2.1. Explicit integration. Kovalevskaya variables

Together with the additional integral S.V. Kovalevskaya has found the remarkable variables that transform the equations of motion (1.1) to Abel–Jacobi form (see [29]). With this form the further integration in  $\theta$ -functions (of two variables) can be performed according to a certain general pattern (see [28]). Here we shall present only the corresponding change of variables.

Kovalevskaya variables  $s_1, s_2$  are defined by the following formulas

$$\begin{aligned} s_1 &= \frac{R - \sqrt{R_1 R_2}}{2(z_1 - z_2)^2}, \quad s_2 = \frac{R + \sqrt{R_1 R_2}}{2(z_1 - z_2)^2}, \\ z_1 &= M_1 + iM_2, \quad z_2 = M_1 - iM_2, \\ R &= R(z_1, z_2) = \frac{1}{4}z_1^2 z_2^2 - \frac{h}{2}(z_1^2 + z_2^2) + c(z_1 + z_2) + \frac{k^2}{4} - 1, \\ R_1 &= R(z_1, z_1), \quad R_2 = R(z_2, z_2), \end{aligned} \quad (2.2)$$

where  $F_1 = (M, \gamma) = c, H = h$ . To simplify the evaluations further we will assume  $x = 1$ .

The equations of motion become

$$\frac{ds_1}{\sqrt{P(s_1)}} = \frac{dt}{s_1 - s_2}, \quad \frac{ds_2}{\sqrt{P(s_2)}} = \frac{dt}{s_2 - s_1}, \quad (2.3)$$

where

$$P(s) = \left( \left( 2s + \frac{h}{2} \right)^2 - \frac{k^2}{16} \right) \left( 4s^3 + 2hs^2 + \left( \frac{h^2}{4} - \frac{k^2}{16} + \frac{1}{4} \right) s + \frac{c^2}{16} \right).$$

Because  $P(s)$  is the polynomial of the fifth degree the quadrature for (2.3) is referred to as *ultraelliptic (hyperelliptic)*.

In paper [BorMam Rhd] we present the generalized Kovalevskaya variables for the similar integrable case on the bundle of brackets containing algebras  $e(3)$ ,  $so(4)$ ,  $so(3, 1)$ .

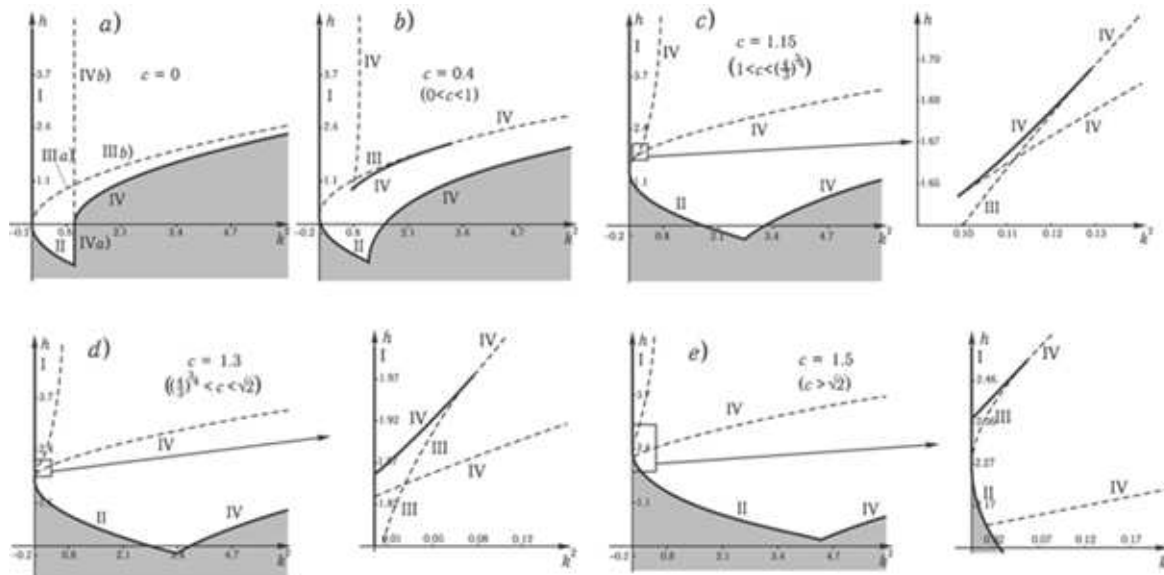


Fig. 2. Bifurcation diagrams of Kovalevskaya case for various  $c$ . Roman numbers denote Appelrot classes. The continuous curves correspond to the stable periodic solutions, dashed — to the unstable ones and to the separatrices.

## 2.2. Bifurcation diagram and Appelrot classes

The values of integrals  $h, c, k$  for which polynomial  $P(s)$  has multiple roots determine on the common space of these integrals the *bifurcation diagram* the collection of two-dimensional surfaces, on which the type of motion changes (see Fig.2). At the same time the ultraelliptic quadratures in (2.3) transform

to the elliptic ones, and the corresponding (particularly special) motions are called *Appelrot classes* [5]. Different branches of the bifurcation diagram correspond to various Appelrot classes.

It is easy to show (and it is a well-known fact) that the Appelrot classes defined by the multiplicity of the roots of polynomial  $P(s) = 0$  coincide with the set of *special Liouville tori* on which integrals  $H, F_1, F_2, F_3$  are dependent, i. e. the Jacobi matrix rank  $\left\| \frac{\partial(H, F_2, F_3, F_4)}{\partial(\mathbf{M}, \gamma)} \right\|$  drops [23]. It is obvious that these special tori in the phase space of the reduced system (i. e. for Euler–Poisson equations) determine the stable and unstable periodic motions and their asymptotic trajectories.

The stability of branches is indicated on the bifurcation diagram presented in Fig.2. Combined with the above described Poincaré phase sections, the diagram is very useful for the study of dynamics because it allows the visual understanding of qualitative behavior of all trajectories of an integrable system in the phase space.

The explicit solutions for Appelrot classes can be obtained directly without equations (2.3). Their construction, involving nonobvious transformations, was started by Appelrot himself [4], and it was obtained in the most complete form by mechanician A. I. Dokshevich [12] from Donetsk. Let's present a part of his results, related mainly to the periodic and asymptotic motions (the most important for dynamics), and try to clarify their mechanical sense.

There are four Appelrot classes.

**I. Delone solution** [11]: Here  $k^2 = 0$ ,  $h > c^2$  and two additional invariant relations appear

$$\frac{M_1^2 - M_2^2}{2} + x\gamma_1 = 0, \quad M_1 M_2 + x\gamma_2 = 0, \quad (2.4)$$

which define the periodic solution of Euler–Poisson equations.

It turns out that the motion in this case at the zero value of area integral  $c = 0$  is periodic not only for the reduced system (on the Poisson sphere), but in the absolute space [17] as well (see Figs. 7-10).

To derive the explicit quadrature, we express all the variables on the common level of integrals and invariant relations (2.4) as the functions of  $M_1$

$$\begin{aligned} M_2^2 &= 2z - M_1^2, \quad M_3^2 = h - M_1^2 \\ x\gamma_1 &= -M_1^2 + z, \quad x\gamma_1 = -M_1(2z - M_1^2)^{1/2}, \quad x\gamma_3 = (x^2 - z^2)^{1/2} \\ z &= \frac{M_1^2 + M_2^2}{2} = (\gamma_1^2 + \gamma_2^2)^{1/2} = x \frac{-cM_1 \pm \sqrt{(h - c^2)(h - M_1^2)}}{h}. \end{aligned} \quad (2.5)$$

Then we obtain the quadrature for  $M_1$

$$\dot{M}_1 = M_2 M_3 = ((h - M_1^2)(2z - M_1^2))^{1/2}, \quad (2.6)$$

which is elliptic at  $h = c^2$ . For  $c = 0$  it is also possible to obtain a simpler explicit solution, if we use variable  $M_3$  instead of  $M_1$ .

It follows from Fig. 2 that under magnification of  $c$  up to  $c = \left(\frac{3}{4}\right)^{3/4}$  branch IV of Appelrot class “runs” into Delone solution and under further magnification up to  $c^2 < 2$ , the branch divides it into three parts. For  $c^2 = 2$ , the branches of all four Appelrot classes merge in point  $h = 2$ ,  $k^2 = 0$ . To the point of their intersection correspond the unstable fixed point on the Poisson sphere (*the Staudé rotation*) ([9]) and to the one-dimensional motion asymptotic to this point, which is easily calculated from (2.6) in elementary functions

$$M_1 = \sqrt{2x} \frac{3 + \operatorname{ch}^2 u \pm 4 \operatorname{ch} u}{9 - \operatorname{ch}^2 u}, \quad u = 2\sqrt{x}t. \quad (2.7)$$

For  $c^2 > 2$ , one branch of class IV also “runs” into Delone solution, while its other branch intersects the part of a parabola, corresponding to the class II.

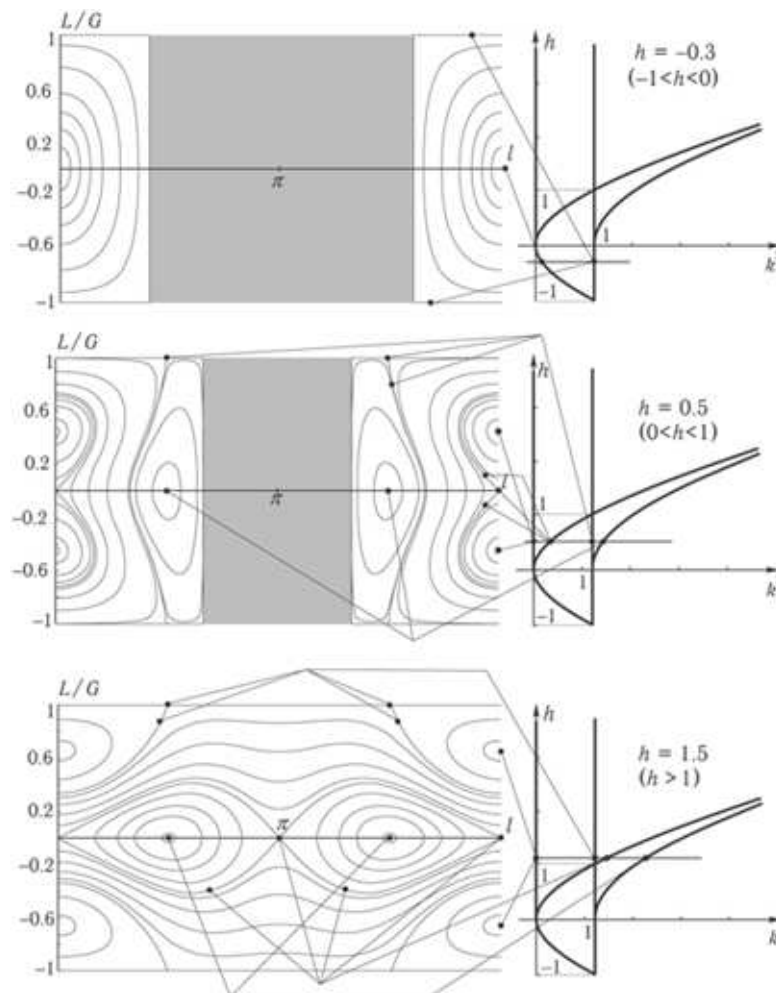


Fig. 3. Phase portraits (sections by plane  $g = \pi/2$ ) for Kovalevskaya case at the zero value of area integral  $c = 0$  (three qualitatively different types are shown). One can see reorganizations of the portraits and the bifurcations of the periodic solutions, which happen during the intersection of critical levels of energy  $h = 0$  and  $h = 1$ . (The gray color fills the nonphysical range of values of  $l$ ,  $L/G$  for the given values of integrals  $h$ ,  $c$ .)

**II.** The solutions of the second class are on the lower branch of parabola  $(h - c^2)^2 = k^2$ , note that  $\frac{1}{2}c^2 - 1 \leq h \leq c^2$ . For  $c = 0$ , the stable periodic trajectories belong to this class, and the rigid body performs flat oscillations in the meridional plane passing through the center of mass, and the conditions  $M_1 = M_3 = 0$ ,  $\gamma_2 = 0$  hold.

For  $c \neq 0$ , there are additional invariant relations

$$M_3 = c\gamma_3, \quad M_1^2 + M_2^2 + \frac{M_1}{c} = k, \quad (2.8)$$

and the explicit integration is presented in [12]. Starting from  $c > \sqrt{2}$ , the branches of classes II and IV begin to intersect.

**III.** The branch of the parabola above the tangency point with the axis  $k^2 = 0$  corresponds to this class. It obeys the conditions

$$(h - c^2)^2 = k^2, \quad c^2 \leq h \leq c^2 + \frac{1}{2c^2}. \quad (2.9)$$

For  $c = 0$ , these requirements determine the whole upper branch of the parabola, and for  $c \neq 0$  this branch is bounded from above by one of the branches of class IV.



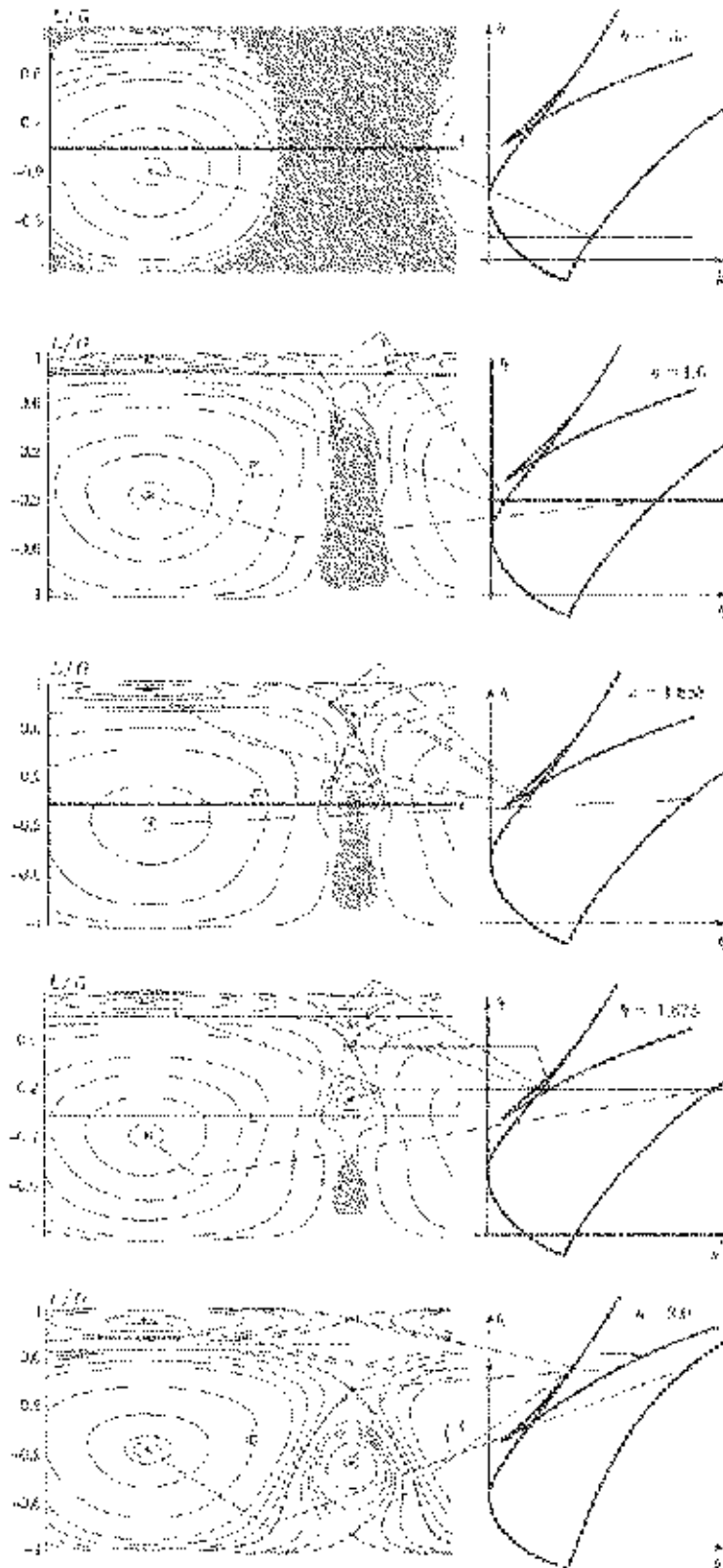


Fig. 4. Phase portraits (sections by plane  $g = \pi$ ) for Kovalevskaya case for  $c = 1.15$  and for the fixed values of energy  $h$ , which correspond to the phase portraits of qualitatively various types. The variables  $l$  and  $L/G$  correspond to the cylindrical involute of the sphere and the phase portrait is symmetrical in relation to the meridian  $l = \pi/2, \frac{3}{4}\pi$ . (The bifurcation diagrams on the right figures are drawn roughly and not to scale.)

Physically, class III corresponds to the unstable periodic solutions and to the solutions asymptotic to them. For  $c = 0$ , the periodic motion for the part of the branch denoted as III a) is oscillations of a physical pendulum in the meridional plane passing through the center of masses, and for the part III b) it is rotations in the same plane. These solutions meet in the point  $h = 1$ , which is the upper unstable equilibrium. Its instability can be strictly proved by various approaches [36]. Later this proof will be obtained by explicit construction of the asymptotic solution.

Let's use the following parametrization of the common level of the integrals of motion corresponding to the third Appelrot class for the zero value of area integral  $c = 0$  [12]

$$\begin{aligned} M_1 &= \sqrt{M_1^2 + M_3^2} \sin \varphi, & M_3 &= \sqrt{M_1^2 + M_3^2} \cos \varphi \\ k_1 &= k \cos 2\theta, & k_2 &= k \sin 2\theta, \end{aligned} \quad (2.10)$$

where  $k_1 = \gamma_1 + \frac{M_1^2 - M_2^2}{2}$ ,  $k_2 = \gamma_2 + M_1 M_2$  (at  $x = 1$ ), Kovalevskaya integral having the form  $k_1^2 + k_2^2 = k^2$ .

Differentiating  $\varphi$  with respect to time, we obtain

$$\dot{\varphi} = M_2 - \frac{M_1 k_2}{M_1^2 + M_3^2}. \quad (2.11)$$

After one more differentiation (2.11) and elimination of  $M_2$  with the help of (2.11), taking into account  $h = k > 0$ , we have the equality

$$2\ddot{\varphi} \cos \varphi + \dot{\varphi} \sin \varphi = 2h \cos^2 \varphi \sin \varphi. \quad (2.12)$$

Multiplying (2.12) by  $\frac{\dot{\varphi}}{\cos^2 \varphi}$  and integrating with respect to time, we obtain

$$\frac{\dot{\varphi}^2}{\cos \varphi} + 2h \cos \varphi = c_1 = \text{const.}$$

The integration constant is obtained from the condition  $\varphi = 0$ , which imply that  $M_1 = 0$ ,  $\dot{\varphi} = M_2$ , and therefore  $c_1^2 = 4x^2$ . Thus,

$$\dot{\varphi}^2 = 2(x - k \cos \varphi) \cos \varphi, \quad k > 0. \quad (2.13)$$

REMARK. For  $c \neq 0$ , we obtain equation [12] for a similar (but a little different) angular variable

$$\dot{\varphi}^2 = 2(x - (k + c^2) \cos \varphi) \cos \varphi.$$

For angle  $\theta$ , we have the equation

$$\dot{\theta} = -M_3 = -\sqrt{M_1^2 + M_3^2} \cos \varphi,$$

which after taking into account integral of energy  $M_1^2 + M_3^2 - k_1 = h$  and condition  $h = k$  resulting in equality  $\sqrt{M_1^2 + M_3^2} = \pm \sqrt{2k} \cos \theta$ , is reduced to the following form:

$$\dot{\theta} = \sqrt{2k} \cos \varphi \cos \theta.$$

After substitution  $\cos \theta = (\text{ch } u)^{-1}$  we can write it as

$$\dot{u} = \sqrt{2k} \cos \varphi.$$



Thus, the complete system of equations defining the asymptotic trajectories of Appelrot class III under condition  $c = 0$ ,  $h = k > 0$  is reduced to the form

$$\begin{aligned} 2\dot{\zeta} &= (1 - \zeta^2)(x - k + (x + k)\zeta^2), \quad \zeta = \operatorname{tg} \frac{\varphi}{2} \\ \dot{u} &= \sqrt{2k} \cos \varphi, \quad \operatorname{ch} u = (\cos \theta)^{-1}. \end{aligned} \quad (2.14)$$

Its solutions are:

1.  $k < x$ ,  $\zeta = \operatorname{cn}(\sqrt{x}t, k_0)$ ,  $k_0^2 = \frac{x+k}{2x}$
2.  $k > x$ ,  $\zeta = \operatorname{dn}\left(\sqrt{\frac{x+k}{2}}t, k_0\right)$ ,  $k_0^2 = \frac{2x}{x+k}$ ,
3.  $k = x$ ,  $\zeta = (\operatorname{ch} \sqrt{x}t)^{-1}$ ,

where  $k_0$  is the absolute value of the corresponding Jacobi elliptic functions.

Using 1–3, it is possible to show, that  $\dot{u}$  is a function of fixed sign, i. e. these solutions in cases 1–2 describe the motions asymptotic to the periodic solution, and in case 3 describe the motions asymptotic to the fixed point. (The analytical quadratures in the case  $c \neq 0$  are more cumbersome [12].)

**IV.** This class consists of two branches (see Fig. 2), one of which corresponds to stable periodic motions, and another — to unstable motions and to separatrices. For  $c = 0$ , these branches meet in point  $k^2 = x^2 = 1$ ,  $h = 0$ .

For  $c \neq 0$ , the parametric equations of the branches are

$$k^2 = 1 + tc + \frac{t^4}{4}, \quad h = \frac{t^2}{2} - \frac{c}{t}, \quad (2.15)$$

$$\begin{aligned} t &\in (-\infty, 0) \cup (c, +\infty), \quad \text{for } c > 0, \\ t &\in (-\infty, +\infty) \setminus \{0\}, \quad \text{for } c < 0, \end{aligned}$$

For  $c = 0$

1.  $k^2 = x^2$ ,  $h < 0$ ,  $h^2 = k^2 + x^2$  (branch IVa);
2.  $k^2 = x^2$ ,  $h > 0$  (branch IVb).

The stable and unstable periodic solutions for Appelrot class IV in the Kovalevskaya case (as well as in a more general case, when the tensor of inertia has the form  $\mathbf{I} = \operatorname{diag}(1, a, 2)$ ,  $a = \operatorname{const}$ , and the solution does not depend on  $a$ ) were found by D. K. Bobylev [7] and V. A. Stekloff [38] (see also [9]).

**Bobylev–Stekloff solution.** For this solution the following relations always hold

$$M_2 = 0, \quad M_1 = m = \operatorname{const},$$

which allow to express  $\gamma$  as a function of  $M_3$

$$\gamma_1 = \frac{c}{m} - M_3^2, \quad \gamma_2 = \left(k^2 - \left(\frac{1}{2}m^2 - \frac{c}{m} + M_3^2\right)^2\right)^{1/2}, \quad \gamma_3 = mM_3$$

and to obtain an elliptic quadrature for  $M_3$

$$\dot{M}_3 = -\left(k^2 - \left(\frac{1}{2}m^2 - \frac{c}{m} + M_3^2\right)^2\right)^{1/2}. \quad (2.16)$$

Here  $h$  and  $k^2$  are defined by parametric equations

$$h = \frac{1}{2}m^2 - \frac{c}{m}, \quad k^2 = 1 + \frac{1}{2}m^4 + cm,$$

i. e. they coincide with (2.15). For  $c = 0$ , the motions occur in the fourth class that correspond to oscillations and rotations obeying the law of physical pendulum in the equatorial plane of the inertia ellipsoid. For these solutions,

$$M_1 = m = 0, \quad \gamma_3 = 0, \quad \dot{M}_3 = -(1 - (h - M_3^2)^2)^{1/2}.$$

The asymptotic solutions for arbitrary values of  $c \neq 0$  can be found in [12], but they are very cumbersome. Let's specify these solutions under the additional conditions

$$k^2 = x^2, \quad h > 0, \quad c = 0. \quad (2.17)$$

For this purpose we use an interesting involute transformation,  $(\mathbf{M}, \boldsymbol{\gamma}) \mapsto (\mathbf{L}, \mathbf{s})$  (its square is equal to identity), found by A. I. Dokshevich:

$$\begin{aligned} L_1 &= -\frac{M_1}{M_1^2 + M_2^2}, & s_1 &= -\gamma_1 + 2x\gamma_3^2 \frac{M_1^2 - M_2^2}{(M_1^2 + M_2^2)^2}, \\ L_2 &= -\frac{M_2}{M_1^2 + M_2^2}, & s_2 &= -\gamma_2 + 4x\gamma_3^2 \frac{M_1 M_2}{(M_1^2 + M_2^2)^2}, \\ L_3 &= M_3 + 2x\gamma_3 \frac{M_1}{M_1^2 + M_2^2}, & s_3 &= \frac{\gamma_3}{M_1^2 + M_2^2}. \end{aligned} \quad (2.18)$$

In new variables  $(\mathbf{L}, \mathbf{s})$  the equations of motion have the form

$$\begin{aligned} \dot{L}_1 &= L_2 L_3, & \dot{s}_1 &= 2L_3 s_2 - 4(k^2 - x^2)s_3 L_2, \\ \dot{L}_2 &= -L_1 L_3 - x s_3, & \dot{s}_2 &= -2L_3 s_1 + 4(k^2 - x^2)s_1 L_3, \\ \dot{L}_3 &= -2x c L_2 + x s_2, & \dot{s}_3 &= s_1 L_2 - s_2 L_1. \end{aligned} \quad (2.19)$$

Under condition (2.17) in system (2.19) the equations for  $L_3, s_1, s_2$  are separated and are reduced to quadratures

$$\begin{aligned} s_2 &= (1 - s_1^2)^{1/2}, & L_3 &= (h + x s_1)^{1/2}, \\ \dot{s}_1 &= 2\sqrt{(h + x s_1)(1 - s_1^2)}. \end{aligned} \quad (2.20)$$

To obtain the solution of complete system (2.19) it suffices to find the solution of a linear second-order equation with coefficients explicitly time-dependent

$$\begin{aligned} L_1 &= s_1^{-1}(-L_3 s_3 \mp s_2 \sqrt{h s_3^2 - \frac{1}{4x} s_1}), & L_2 &= \sqrt{h s_3^2 - \frac{1}{4x} s_1}, \\ \ddot{s}_3 &= -x(1 + 2s_1)s_3. \end{aligned} \quad (2.21)$$

Equations (2.20), (2.21) describe the solutions asymptotic to periodic motions under conditions (2.17) (see Fig. 15).

For  $h = x$ , which corresponds to the energy of the upper unstable equilibrium, we shall obtain one more (in addition to class III) solution asymptotic to the equilibrium expressed in elementary functions

$$s_1 = 1 - 2 \operatorname{th} u, \quad s_2 = 2 \frac{\operatorname{th} u}{\operatorname{ch} u}, \quad L_3 = -\frac{\sqrt{2x}}{\operatorname{ch} u}, \quad u = \sqrt{2x}t.$$

Appelrot classes define the most simple motions both in reduced, and in absolute phase space. The other motions of Kovalevskaya top have quasiperiodic character and depend on the corresponding domain of the bifurcation diagram. Under perturbation of the Kovalevskaya case, a stochastic zone appear near the unstable solutions and their separatrices. Unfortunately, the (asymptotic) solutions presented in this paragraph do not yet allow (because of various reasons) to advance in the analytical investigation of nonintegrability of the perturbed Kovalevskaya top (the proof of nonintegrability for  $c = 0$  is obtained by variational methods in [8]).

### 2.3. Phase portrait and visualization of particular special solutions

For each fixed value of area integral  $(M, \gamma) = c$ , defining various types of the bifurcation diagrams on the plane  $(k^2, h)$ , there is its own collection of phase portraits. Fixing the level of energy  $h$  we obtain several various types of phase portraits, which are defined by intersections of straight line  $h = \text{const}$  with the bifurcation diagram. Here we present two sets of the phase portraits corresponding to the most simple ( $c = 0$ , Fig. ??) and to the most complicated ( $1 < c < (\frac{4}{3})^{3/4}$ , Fig. ??) bifurcation diagrams. Also, the form of some “particular special” solutions on the Poisson sphere and in the absolute space is presented in the following paragraphs.

REMARK. The investigation of invariant tori topology with the help of Poincaré sections is also presented in [14], in different variables and without explanation of mechanical meaning of various motions (in particular, without there the analysis of stability).

**Phase portrait for  $c = 0$ .** In this case the bifurcation diagram consists of two parts of parabolas and two straight lines (see Fig. 2a). The physical sense of branches corresponding to the parabola  $h^2 = k^2$  and to the straight line  $k^2 = 1$  is especially clear and is described above. On the parabola there are solutions describing flat oscillations and rigid body rotations in the meridional plane (around axis  $Oy$ , perpendicular to axis  $Ox$ , on which the center of mass is situated), and on the straight line — the one describing flat oscillations and rotations in the equatorial plane (around axis  $Oz$ ). On the remaining branches  $k^2 = 0$  and  $h^2 = k^2 - 1$ , Delone and Bobylev–Stekloff solutions are situated, accordingly.

Above we have presented the phase portraits and indicated where they are situated on the bifurcation diagram. It follows from Fig. 2a) that there are three intervals for the constant of energy  $h$ :  $(-1, 0)$ ,  $(0, 1)$ ,  $(1, \infty)$ , to each of which the qualitatively various types of phase portraits correspond (see Fig. ??).

**Phase portrait for  $c = 1.15$  ( $1 < c < (\frac{4}{3})^{3/4}$ )** . With the bifurcation diagram (Fig. 2c) it is possible to establish that there are five energy intervals with corresponding types of phase portrait (see Fig. ??). In this case the periodic solutions corresponding to the branches of the bifurcation diagram do not have the forms as simple, as for  $c = 0$ , though they tend to it at  $h \gg c$ .

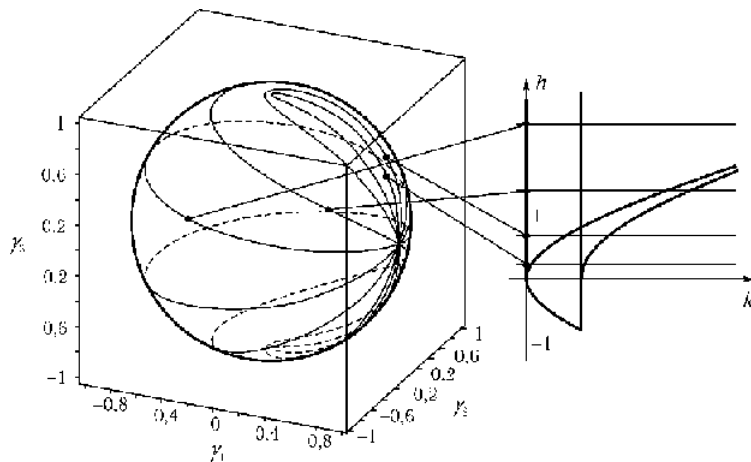


Fig. 5. Delone solution. Motion of the unit vector  $\gamma$  for the zero value of area integral ( $c = 0$ ) and for various values of energy.

REMARK. For the construction of the phase portraits we use Poincaré sections and Andoyer–Deprit variables. For  $c = 0$  we choose  $g = \frac{\pi}{2}$  as an intersecting plane, and for  $c = 1.15$  we choose  $g = \pi$ . We change

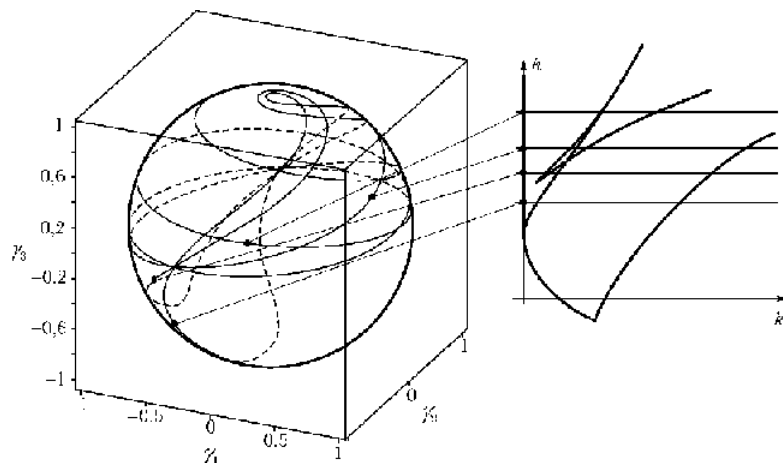


Fig. 6. Delone Solution. Motion of the unit vector  $\gamma$  for the nonzero value of area integral ( $c = 1.15$ ) and for various values of energy  $h$ .

the intersecting plane because in this case not all periodic solutions intersect plane  $g = \frac{\pi}{2}$ . Let's also mention the different types of symmetry of phase portraits on sphere ( $l, L/G$ ): for  $g = \frac{\pi}{2}$  the portrait is symmetrical with respect to the equator (axis  $L/G = 0$ ), and for  $g = \pi$  it is symmetrical with respect to the meridional plane ( $l = \frac{\pi}{2}, \frac{3}{2}\pi$ ).

Let's proceed to visualization of some most interesting motions of a rigid body in the reduced and absolute spaces.

**Delone solution ( $k^2 = 0$ ).** In this case the trajectory of the vertical unit vector  $\gamma$  on the Poisson sphere is represented by curves with the figure-of-eight type (see Fig. 5, 6), and for  $c = 0$  (Fig. 5) the points of self-intersection of these "figure-of-eight type curves" coincide and have coordinates  $\gamma = (1, 0, 0)$ . This point determines the lower position of the center of mass of a rigid body. When  $c$  increases the irregular "figure-of-eight type curves" also appear on the Poisson sphere, all of them have the same two points of intersection on the equator of the Poisson sphere (see Fig. 6).

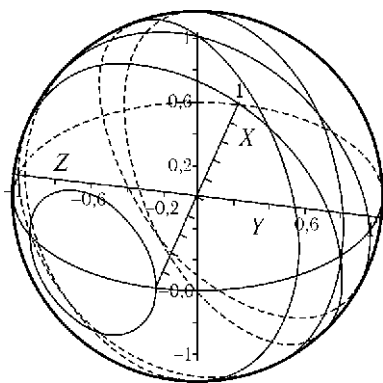


Fig. 7. Delone solution. Motion of the apexes in the fixed frame of reference for the zero value of area integral ( $c = 0$ ).

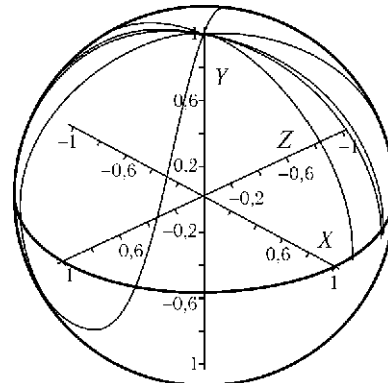


Fig. 8. Delone solution. Motion of the apex of the center of mass for  $c = 0$  and various  $h$ .

It is known that for  $c = 0$  Delone solution determines the periodic motions in both the reduced system and the absolute space [18]. For  $c \neq 0$  this assertion is not valid and the motion of a rigid body in the absolute space is quasi-periodic. In figures 7–10 the trajectories of three apexes of a rigid

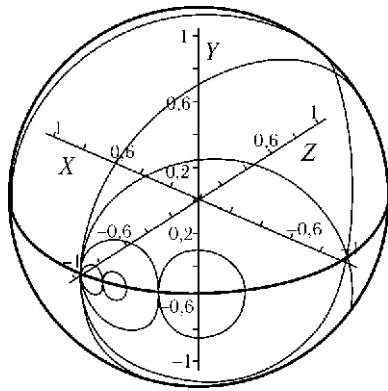


Fig. 9. Delone solution. Motion of the apex situated in the equatorial plane perpendicular to the radius-vector of the center of mass for  $c = 0$  and various  $h$ .

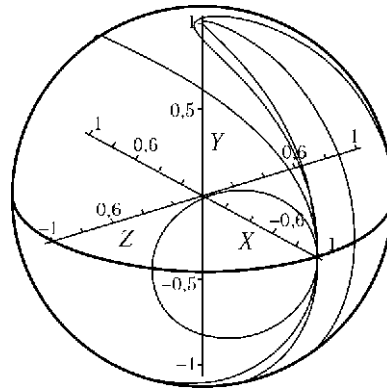


Fig. 10. Delone solution. Motion of the apex of the axis of dynamic symmetry for  $c = 0$  and various  $h$ .

body are shown for  $c = 0$  and for various values of energy. In all figures the fixed axes  $OXYZ$  are arbitrarily rotated to show the best view of obtained trajectories.

**Bobylev–Stekloff solution.** Bobylev–Stekloff solution on the bifurcation diagram (see Fig. 2) is situated on the lower right branch and corresponds to the stable periodic solution on the Poisson sphere (see Fig. 11, 12).

It is clearly visible in Fig. 11, that for  $c = 0$  all trajectories on the Poisson sphere pass through the points of its equator  $(0, 1, 0)$  and  $(0, -1, 0)$ , not intersecting the meridional plane  $\gamma_1 = 0$ . The remarkable motion of the center of mass in absolute space corresponds to this case: *the center of mass describe the curves with cusps, which for all energies are situated on the equator* (see Fig. 13).

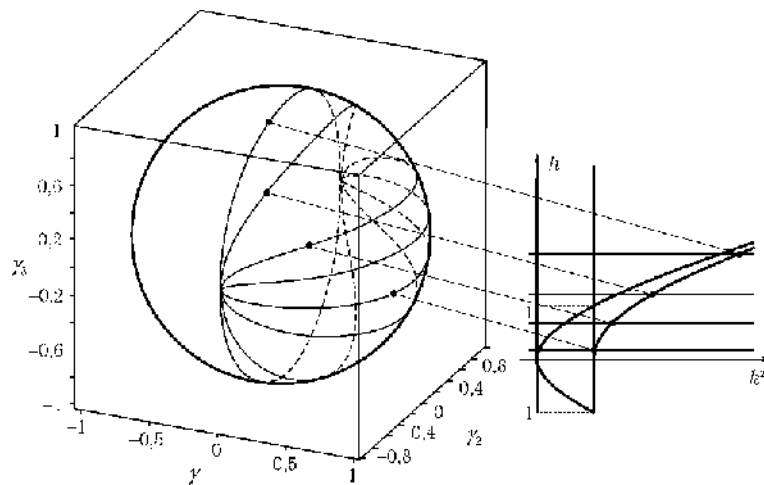


Fig. 11. Bobylev–Stekloff solution. Motion of the vertical unit vector on the Poisson sphere for  $c = 0$  and various values of energy.

The trajectories on the Poisson sphere for  $c \neq 0$  are presented in Fig. 12, in this case the apex of the center of mass traces in the absolute space the curves with cusps on the same latitude, which depends on a constant value of energy  $h$  (see Fig. 14). Physically, Bobylev–Stekloff solution can be implemented as follows: a body is twisted around the axis passing through the center of mass and arbitrarily positioned in the absolute space, then it is released without an initial impulse.

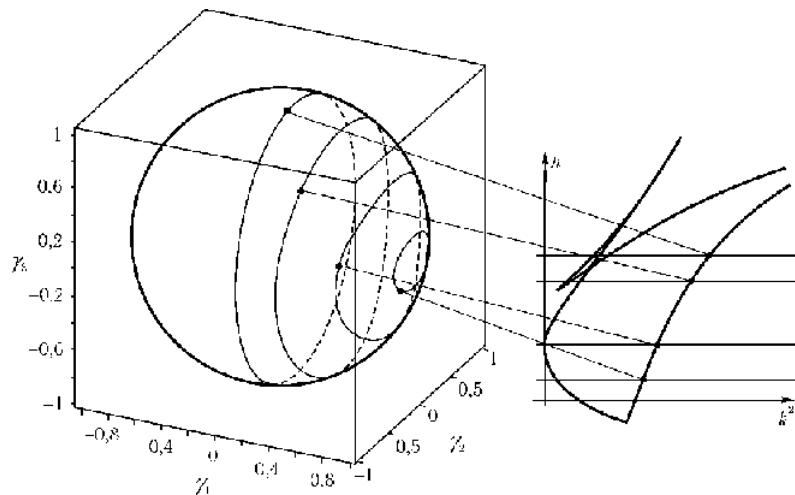


Fig. 12. Bobylev–Stekloff solution. Motion of the vertical unit vector on the Poisson sphere for  $c \neq 0$  ( $c = 1.15$ ) and various values of energy.

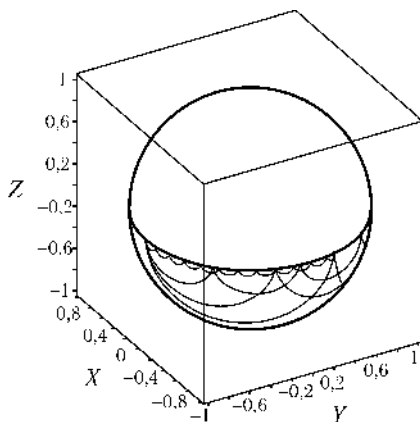


Fig. 13. Bobylev–Stekloff solution. Motion of the apex passing through the center of mass in the absolute space for  $c = 0$  and various  $h$ .

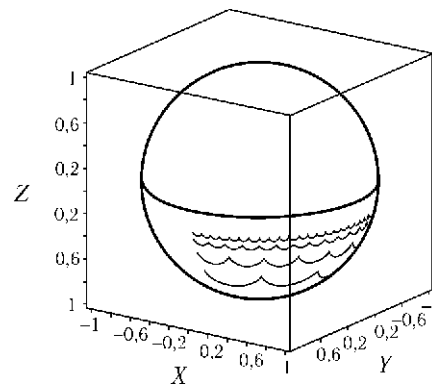


Fig. 14. Bobylev–Stekloff solution. Motion of the apex passing through the center of mass in the absolute space for  $c \neq 0$  ( $c = 1.15$ ) and various  $h$ .

REMARK 2. Motion of the remaining apices in the absolute space is very complicated, therefore we do not present it.

**Unstable periodic solutions and the separatrices** for Kovalevskaya case have very complicated form both on the Poisson sphere, and in the absolute space. In Fig. 15, the trajectories of motion corresponding to the separatrices for  $c \neq 0$  ( $c = 1.15$ ) and for the same value of energy  $h = 2$  are presented. It is clearly visible that most of the time the trajectory is staying near the periodic solution, in the figure this is shown by darker shading.

These trajectories in some sense represent the complexity of Kovalevskaya integrable case, some motions in this case have visually chaotic character (in the absolute space the motion looks even more irregular).

REMARK 3. Let's give one more representation of Kovalevskaya integral, this time as a sum of squares. For this purpose we use the projections of the moment on semimoving axes

$$S_1 = M_1\gamma_1 + M_2\gamma_2, \quad S_3 = M_1\gamma_2 - M_2\gamma_1.$$



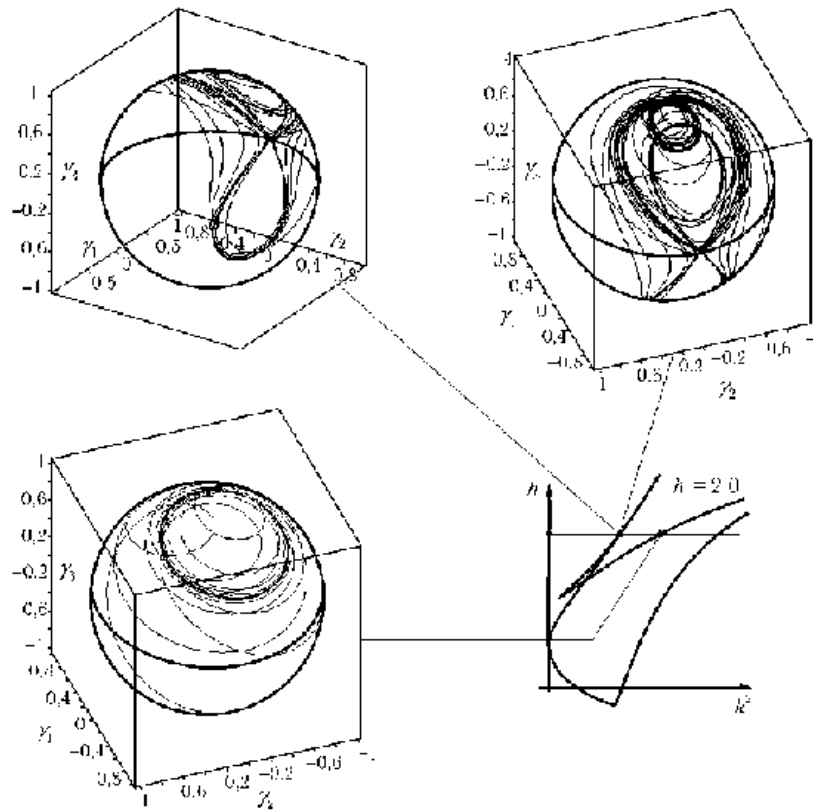


Fig. 15. Trajectories on the Poisson sphere for the solutions asymptotical to the unstable periodic solutions.

It is possible to show that we can write Kovalevskaya integral in the form

$$F = \left( \frac{M_1^2 + M_2^2}{2} \right)^2 + x(M_1 S_1 + M_2 S_2) + x^2(\gamma_1^2 + \gamma_2^2).$$

Letting  $\mathbf{S} = (S_1, S_2)$  and  $\widetilde{\mathbf{M}} = (M_1, M_2)$  be two-dimensional vectors, we denote the angle between them as  $\lambda$  (see Fig. 16). Taking into account, that  $\gamma_1^2 + \gamma_2^2 = \sin^2 \theta$ , where  $\theta$  is the angle between the vertical axis and the symmetry axis of the inertia ellipsoid, we can write Kovalevskaya integral in the form

$$F = \frac{1}{4} G^4 \sin^2 \lambda + \left( \frac{G^2 \cos \lambda}{2} + x \sin \theta \right)^2 = k^2, \quad G^2 = M^2.$$

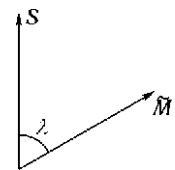


Fig. 16

REMARK 4. Let's also present the interesting nonlinear transformation preserving the structure of algebra  $so(3)$ :

$$K_1 = \frac{M_1^2 - M_2^2}{2\sqrt{M_1^2 + M_2^2}}, \quad K_2 = \frac{M_1 M_2}{\sqrt{M_1^2 + M_2^2}}, \quad K_3 = \frac{1}{2} M_3.$$

It is possible to show that, for the system Adoyer–Deprit canonical variables, the transformation corresponds to the canonical transformation of the type  $(L, l) \mapsto \left( \frac{L}{2}, 2l \right)$ .

## 2.4. The historical comments

**Kovalevskaya method.** When S. V. Kovalevskaya discovered the general case of integrability she was not guided by physical reasons, instead she developed the ideas of K. Weierstrass, P. Painlevé and H. Poincaré concerning the investigation of the analytic continuation of the solutions of a system of ordinary differential equations into the complex plane of time. S. V. Kovalevskaya assumed that in

integrable cases the general solution on the complex plane has no other singularities, except for poles. This assumption allowed to obtain the conditions when the additional integral exists. In addition to the determination of the first integral, S. V. Kovalevskaya found the quite nontrivial system of variables, in which the equations have the Abel–Jacobi form, and besides she obtained the explicit solution in  $\theta$ -functions. The reduction to quadratures in Kovalevskaya case is still considered to be very complicated and does not allow any essential simplification.

A. M. Lyapunov in paper [31] improved Kovalevskaya’s analysis, having required for the sake of integrability the single-valuedness (the meromorphic property) of the general solution as a complex function of time, and studying the solutions of variational equation. Lyapunov method is slightly different from Kovalevskaya’s approach, which was further developed in papers by M. Adler, P. van Moerbeke, who associated the presence of the full parametric set of single-valued Laurent (polar) expansions with the algebraic integrability of the system (in some narrow sense [1, 2]). The most complete analysis of the full parametric expansions in Euler–Poisson equations is contained in paper [30]. The classical presentation of Kovalevskaya and Lyapunov results is included in several textbooks [3, 16].

The investigations of Kovalevskaya laid the foundations of a new method of integrability analysis of a system, and at the same time they were the first example of the search of obstructions to integrability that evolved recently into a separate science [26]. Let’s also note that in spite of the fact that there exist certain strict results concerning the relation of the branching of the general solution with the non-existence of first integrals [26], Kovalevskaya method nevertheless remains as a test of integrability. It is ambiguous in many aspects, and its application to various problems requires particular skills and additional arguments. In the physical literature this method is usually referred to as *Painlevé–Kovalevskaya test*.

**Kovalevskaya case, its analysis and generalizations.** A geometrical interpretation of Kovalevskaya case, which is not, however, natural enough and an original method of reduction of Kovalevskaya case to quadratures were suggested by N. E. Zhukovsky [22]. He also used Kovalevskaya variables to construct some curvilinear coordinates on a plane (plane  $M_1$ ,  $M_2$ ), which correspond to the separable variables of Kovalevskaya top. His arguments were simplified by W. Tannenberg and K. Suslov [39, 40].

F. Kötter also simplified slightly the method of explicit integration in Kovalevskaya case [25] and suggested investigation of the motion in a frame of reference, uniformly rotating around the vertical axis. From the modern point of view, the introduction of Kovalevskaya variables and the reduction to Abel equations is presented in [27]. The qualitative analysis of the motion of the axis of dynamic symmetry is presented in [27], the topological and bifurcational analysis is presented in [23]. The action-angle variables for Kovalevskaya top are constructed in [41] (see also [13]). We discuss them in our book [9]. N. I. Mertsalov carried out the natural experiments, but he did not reveal, however, any particular properties of the top’s motion [27, 32].

The structure of complex tori is explored in [6] with the help of algebraic geometry methods. The bifurcation diagrams for Kovalevskaya case in connection with Kolossoff analogy are considered in [15].

The quantization of Kovalevskaya top is a problem which is discussed from the very moment of creation of quantum mechanics (Lapporte, 1933), but still it is not completely solved [24, 34]. In paper [13], the Picard–Fuchs equation, originating from the integration of Kovalevskaya case, was written out. The first Lax representation for the  $n$ -dimensional Kovalevskaya case without a spectral parameter was constructed by A. M. Perelomov [33]. The representation with a spectral parameter in the general formulation (for motion in two homogeneous fields) was suggested by A. G. Rejman and M. A. Semenov–Tyan-Shansky [35]. This generalization of Kovalevskaya case is still poorly investigated (in particular, it is not integrated in quadratures, and lacks topological and qualitative analysis).

REMARK 5. In paper [20], K. P. Hadeler and E. N. Selivanova show the family of systems on sphere  $S^2$ , which allow an integral of the fourth degree with respect to the momentum components, which can not be

reduced to Kovalevskaya case (or to its generalization, described by Goryachev). In paper [37], the similar construction is suggested for the systems with an integral of the third degree. Note only that in these papers no explicit form of an additional integrals was given, and the corresponding family is determined in the result of the solution of a certain differential equation, for which the existence theorems are proved.

### 3. Goryachev–Chaplygin case

Let's consider the particular integrable case of Goryachev–Chaplygin case, with the momentum vector situated on the horizontal plane, i. e.  $(\mathbf{M}, \boldsymbol{\gamma}) = 0$ . It has almost the same restrictions on the dynamical parameters, as Kovalevskaya case, but the ratio of the moments of inertia is now equal to four ( $\frac{a_3}{a_1} = 4$ ), instead of two. The Hamiltonian and the additional integral are written as

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - x\gamma,$$

$$F = M_3(M_1^2 + M_2^2) + xM_1\gamma_3.$$

#### 3.1. Explicit integration

The variables of Kovalevskaya type that reduce the system to Abel–Jacobi equations were found by S. A. Chaplygin [10]. They are determined by formulas

$$M_1^2 + M_2^2 = 4uv, \quad M_3 = u - v \quad (3.1)$$

and satisfy the system of differential equations

$$\frac{du}{\sqrt{P_1(u)}} - \frac{dv}{\sqrt{P_2(v)}} = 0,$$

$$\frac{2u du}{\sqrt{P_1(u)}} + \frac{2v dv}{\sqrt{P_2(v)}} = dt, \quad (3.2)$$

$$P_1(u) = -\left(u^3 - \frac{1}{2}(h-x)u - \frac{1}{4}f\right)\left(u^3 - \frac{1}{2}(h+x)u - \frac{1}{4}f\right),$$

$$P_2(v) = -\left(v^3 - \frac{1}{2}(h-x)v + \frac{1}{4}f\right)\left(v^3 - \frac{1}{2}(h+x)v + \frac{1}{4}f\right),$$

where  $h, f$  are the constant values of energy integral and Chaplygin integral ( $H = h, F = f$ ).

REMARK. Essentially, by introducing variables  $u, v$  Chaplygin constructed the system of Andoyer–Deprit variables, or more precisely, the variables connected with them by the relation  $L = u - v, G = u + v$ . In [9] the generalization of Goryachev–Chaplygin case is constructed with the help of analysis of Andoyer–Deprit variables for a bundle of Poisson brackets, which include algebras  $so(4)$ ,  $e(3)$ ,  $so(3, 1)$ , and the corresponding separable variables are found.

#### 3.2. Bifurcation diagram and phase portrait

Using functions  $P_1(u)$ ,  $P_2(v)$  and the condition of multiplicity of these polynomials' roots it is easy to construct the bifurcation diagram [23]. On the plane  $(f, h)$  it consists of three branches (Fig. ??):

- I.  $f = 0, \quad h > -1,$
- II.  $h = \frac{3}{2}t^2 + 1, \quad f = t^3, \quad t \in (-\infty, +\infty),$
- III.  $h = \frac{3}{2}t^2 - 1, \quad f = t^3, \quad t \in (-\infty, +\infty).$

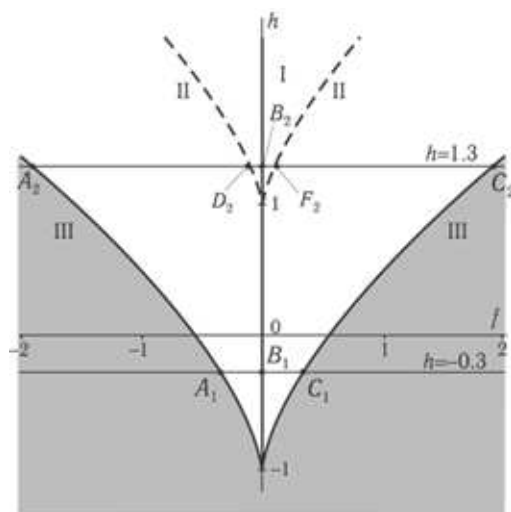


Fig. 17. Bifurcation diagram of Goryachev–Chaplygin case. The nonphysical area of the integrals' values is marked by grey color. Also we indicate two levels of energy, for which the phase portraits are constructed (see Fig. 18, 19). Letters  $A_i, B_i, C_i, \dots$  denote the periodic solutions and separatrices, which are similarly denoted on phase portraits.

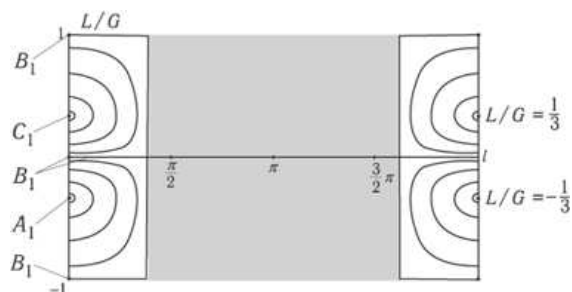


Fig. 18. Phase portrait of Goryachev–Chaplygin case for  $h = 0.3$  (section by plane  $g = \pi/2$ ). Letters  $A_1, B_1, C_1$  denote the periodic solutions situated on the branches of the bifurcation diagram (Fig. 17). Point  $B_1$  on the bifurcation diagram, for which  $f = 0$ , corresponds, at first, to the two pendulum periodic solutions (they are situated on the phase portrait in the poles of sphere  $L/G = \pm 1$  and in point  $l = 0, L/G = 0$ ) and, secondly, to the whole straight line  $L/G = 0, l \neq 0$  that is also filled by periodic solutions (Goryachev solution) of pendulum type.

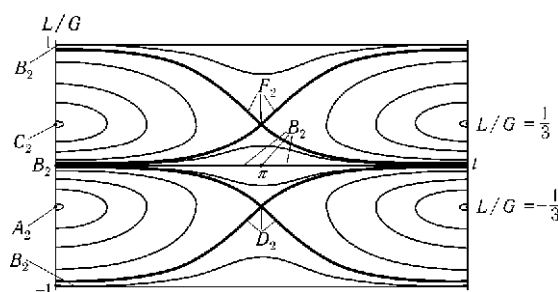


Fig. 19. Phase portrait of Goryachev–Chaplygin case for  $h = 1.3$  (section by plane  $g = \pi/2$ ). Letters  $A_2, B_2, C_2, D_2, F_2$  denote the periodic solutions situated on the branches of the bifurcation diagram (Fig. 17). By contrast with the previous portrait the unstable solutions (and the separatrices to them)  $D_2$  and  $F_2$  are added. Also, same as above, point  $B_2$  on the bifurcation diagram corresponds to four rotational periodic solutions (rotations in equatorial and meridional plane with taking into account the direction of rotation). They are represented by points  $L/G = \pm 1$  and  $l = 0, \pi, L/G = 0$ , and by the whole straight line  $L/G = 0$ , which is filled by periodic solutions (Goryachev solutions) of reduced system.

Three periodic solutions belong to the first class (I):

- 1) Rotations and oscillations in the equatorial plane of the inertia ellipsoid ( $M_1 = M_3 = 0, \gamma_2 = 0$ );
- 2) Rotations and oscillations in the meridional plane of the inertia ellipsoid ( $M_1 = M_2 = 0, \gamma_3 = 0$ );
- 3) The particular Goryachev solutions, corresponding to  $f = 0$ .

Unfortunately, the solutions situated on the branches II, III are practically not investigated at all. The phase portraits corresponding to various values of energy are presented in Fig. 18, 19.

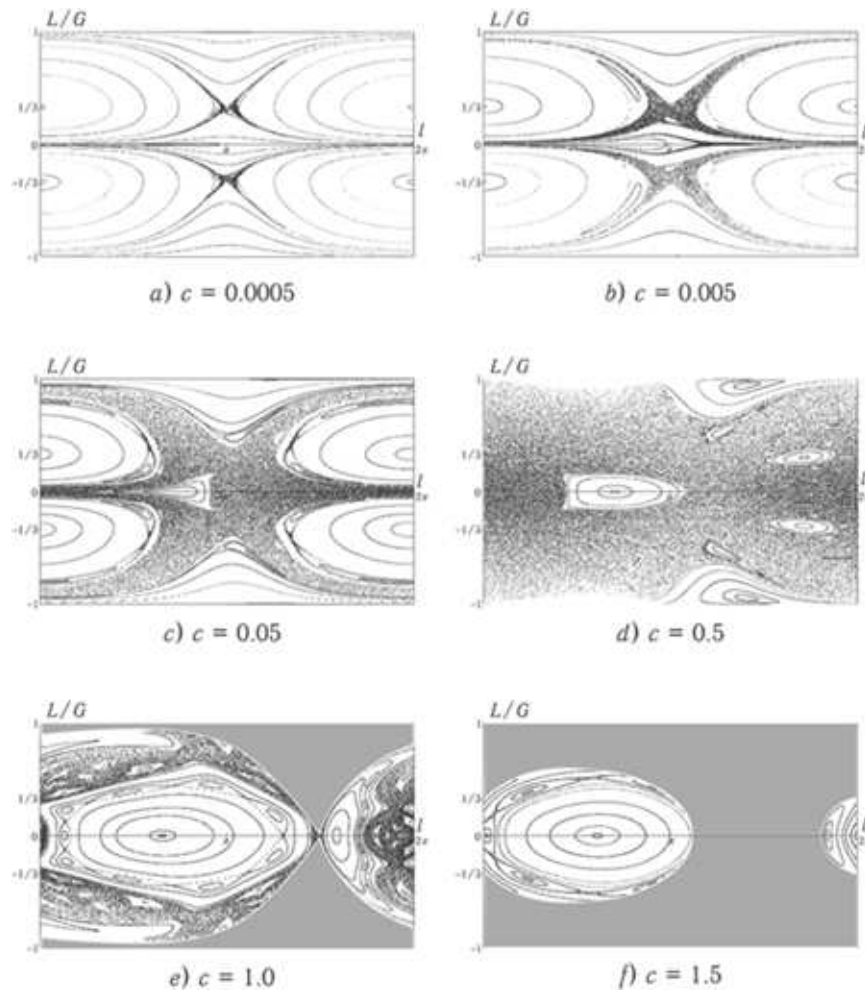


Fig. 20. Perturbation of Goryachev–Chaplygin case for the fixed value of energy ( $h = 1.5$ ) and for the increase of the constant value of area integral (section by plane  $g = \pi/2$ ). The figures show that there is a stochastic layer near separatrices, which at first increases, and then decreases together with the area of possible motion. It is interesting that under further magnification of  $c$ , the area of possible motion decreases together with the stochastic layer up to the complete disappearance.

REMARK 6. The absence of explicit analytical expressions for asymptotic solutions is also an obstruction to investigation of a perturbed system. Note that N. I. Mertsalov in paper [32] has made the statement concerning the integrability of Goryachev–Chaplygin top equations for  $c = (M, \gamma) \neq 0$ . The computer experiments presented in Fig. 23, show that this statement is wrong, and there is a stochastic layer near unstable manifolds for  $c \neq 0$ , which implies nonintegrability.

### 3.3. Visualization of particular special solutions

Among the periodic solutions in Goryachev–Chaplygin problem *Goryachev solution* is very special. On the bifurcation diagram it is situated on straight line  $f = 0$ , besides this line contains the periodic solutions of Euler–Poisson equations that correspond to the oscillations (for  $h < 1$ ) and rotations ( $h > 1$ ) of the rigid body in planes  $Oxy$  and  $Oxz$ , obeying the compound pendulum law. Let's discuss in detail Goryachev solution and the solutions situated on branches II and III (see Fig. 17).

**Goryachev solution.** For this solution there are two additional invariant relations [12]

$$M_1^2 + M_2^2 = bM_1^{2/3}, \quad f = M_3(M_1^2 + M_2^2) + M_1\gamma_3 = 0, \quad (b > 0). \quad (3.3)$$

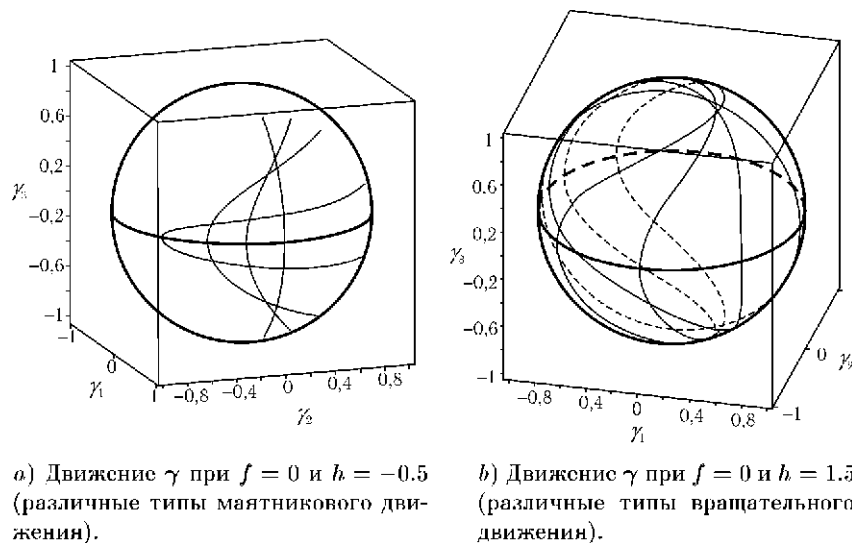


Fig. 21. “Goryachev solution” represents the whole torus, filled by periodic solutions of the reduced system  $(M, \gamma)$  (the so called resonance  $1 : 1$ ); for  $h < 1$  they are pendulum type solutions, and for  $h > 1$  they are rotary type solutions. On this and the following figures the trajectories on the Poisson sphere corresponding to various solutions on this torus are presented.

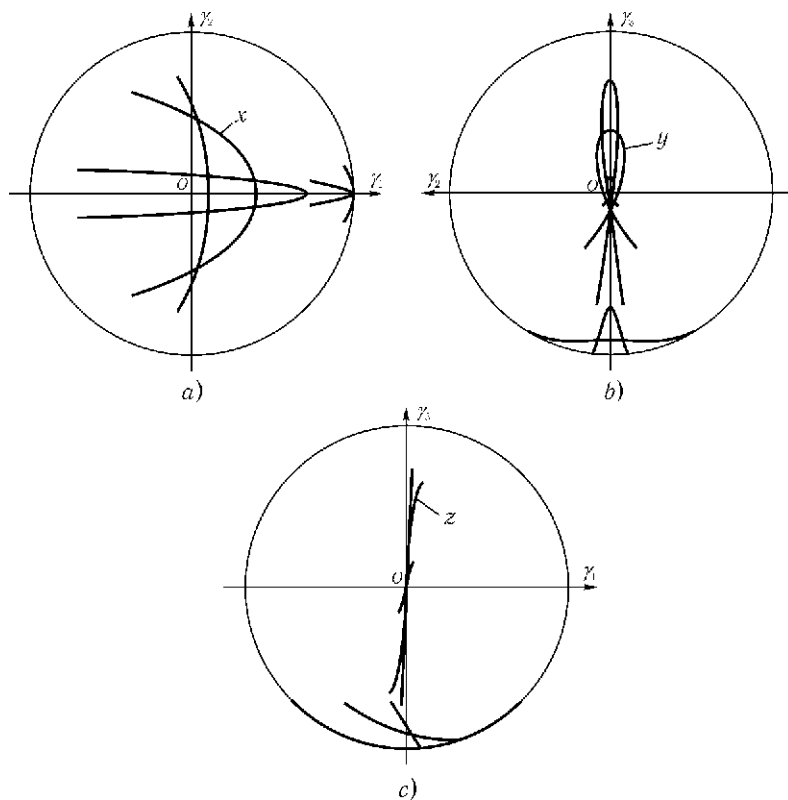


Fig. 22. This figure illustrates the behavior of the principal axes of a rigid body in the fixed frame of reference for Goryachev solutions at a fixed value of energy  $h < 1$  ( $h = -0.7$ ). It can be clearly seen that this solutions are *periodic* in the absolute space, which with the increase of parameter  $b$  changes from oscillations in plane  $Oxy$  to oscillations in plane  $Oxz$ . (Letters  $x, y, z$  denote the axes bound with the body.)



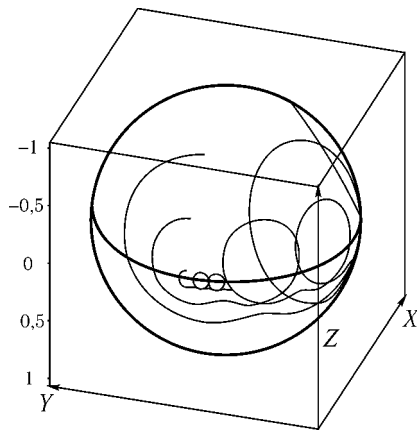


Fig. 23. This figure illustrate the quasi-periodic motion in the absolute space (the motion of the principal axis  $Oy$  is shown) for Goryachev solution for  $h > 1$  ( $h = 1.7$ ).

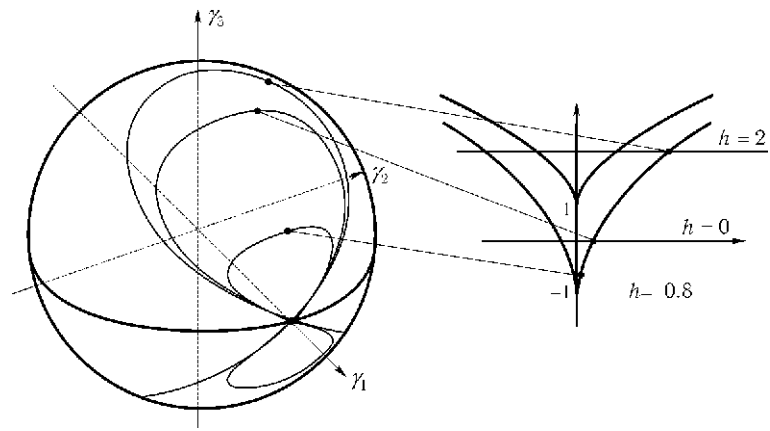


Fig. 24. Motion of the vertical unit vector  $\gamma$  on the Poisson sphere for the stable periodic motion in Goryachev–Chaplygin case for various values of energy.

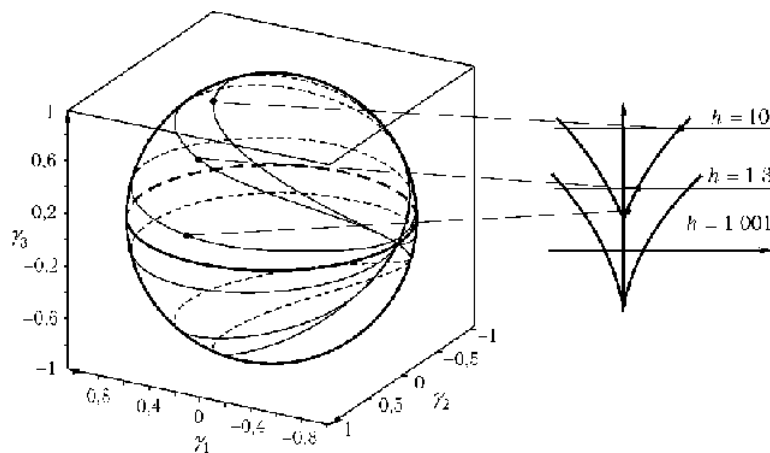


Fig. 25. Motion of the vertical unit vector  $\gamma$  on the Poisson sphere for the unstable periodic solution in Goryachev–Chaplygin case for various values of energy.

An arbitrary constant  $b$  in these relations specifies a parameterization of the whole set of periodic solutions: in the phase space it is a degenerated torus filled by periodic solutions. Relations (3.3) were obtained by D.N. Goryachev, which made S.A. Chaplygin understand that the condition  $f = 0$  is too strong and obtained solution (3.2) in the conventional form. For  $h < 1$  and with increase of  $b$  from 0 up to  $b_{\max}$ , the solution changes from oscillation in the equatorial plane to oscillation in the meridional plane (Fig. 17). On the phase portrait (see Fig. 18) it corresponds to straight line  $L/G = 0$  and to the meridian connecting it with poles. For  $h > 1$  and with the increase of  $b$  from 0 up to  $b_{\max}$  the solution changes from a rotation in the equatorial plane to another one (in the opposite direction, Fig. 19).

The motion of the apex on the Poisson sphere is presented in Fig. 17. The remarkable phenomenon, which was unnoticed earlier, is the fact that for Goryachev solutions in the absolute space for  $h < 1$ , the motion is periodic one of oscillatory type (see Fig. 19). And for  $h > 1$  the corresponding motion is quasiperiodic and bifrequency (Fig. 23).

All indicated facts practically cannot be seen immediately from the analytical solution, which for the first time was obtained by Goryachev in a very cumbersome form [19]. Despite of some

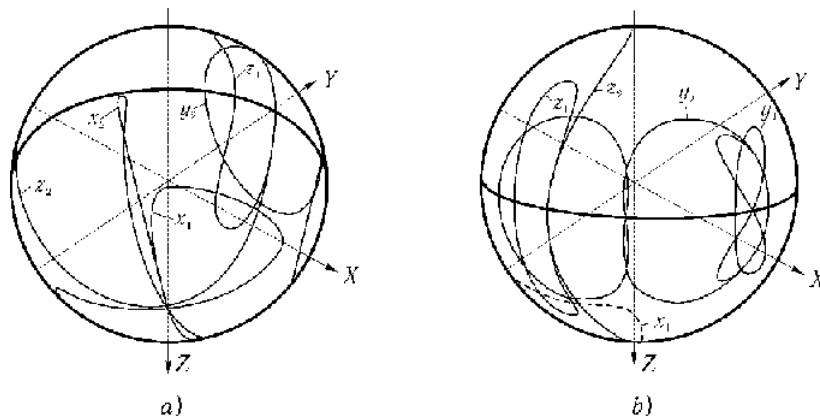


Fig. 26. Motion of the apices of the principal axes of a rigid body in the absolute space in Goryachev–Chaplygin case for the stable periodic solution on branch III in Fig. 17, for two various values of energy  $h_1, h_2$  from the different points of view. Letters  $x_i, y_i, z_i, i = 1, 2$  denote the trajectories of the relevant axes corresponding to the same values of energy.

simplifications available, for example, in [12], the explicit formulas only partially allow to understand the character of the motions, obtained by the computer methods.

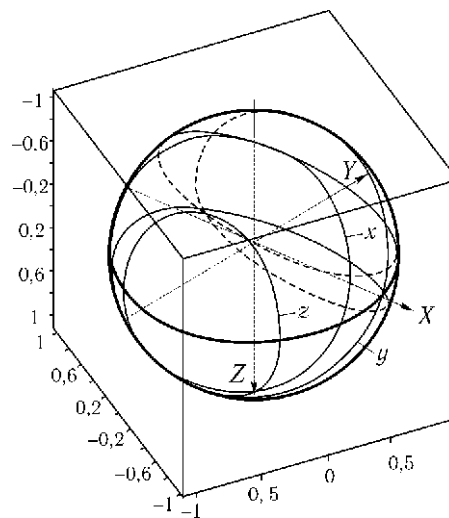


Fig. 27. Motion of the apices of the principal axes of a rigid body in the absolute space in Goryachev–Chaplygin case for the unstable periodic solution on branch II in Fig. 17, for a single value of an energy. Letters  $x, y, z$  denote the trajectories of the corresponding axes. (Motion for other values of energy do not differ qualitatively, therefore we do not present them.)

**The stable and unstable periodic solutions** of Euler–Poisson equations for Goryachev–Chaplygin case are situated in the bifurcation diagram on branches III and II, accordingly (see Fig. 17, 24–??). The numerical investigations show that the motions of the initial system in the absolute space, corresponding to these solutions are *also periodic at any value of energy* (see Fig. 26, ??). This fact, apparently, can not be found in the present literature and reflects the specific character of the rigid body dynamics for the zero value of area integral  $(\mathbf{M}, \boldsymbol{\gamma}) = 0$  (compare with Delone solutions for Kovalevskaya case, [9]). Instead of the formal proof we present a series of figures visually confirming this statement. On them the trajectories of system are presented both on the Poisson sphere and as the trajectories of the apices in the absolute space, the majority of them are complicated enough.

The general conclusion about Goryachev–Chaplygin case is the observation that in its analysis we deal with interesting oscillatory (rotary) motions in the absolute space, i. e. it is possible to speak about a certain complicated pendulum. However, the area of application of such oscillations, is not very clear yet. Note also the comparative simplicity of Goryachev–Chaplygin top's motions in comparison with those of Kovalevskaya top. Few analytical results obtained by study of Goryachev–Chaplygin case cannot give the visual representation of the motion. On the contrary, the computer investigation of the motion discovers its remarkable properties that are also typical for the related integrable systems.

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